

## Square Roots: Introduction & Simplification

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You already know about squaring. For instance,  $2^2 = 4$ ,  $3^2 = 9$ , etc. The backwards of squaring is square-rooting. The symbol for square-rooting is " $\sqrt{\quad}$ ", the "radical" symbol. It is used like this:

$$2^2 = 4, \text{ so } \sqrt{4} = 2$$

$$3^2 = 9, \text{ so } \sqrt{9} = 3$$

Now, you can take any counting number and square it, and end up with a nice neat number. But it doesn't work going backwards. Think about  $\sqrt{3}$ . There is no nice neat number that squares to 3. Then  $\sqrt{3}$  can be handled in either of two ways. If you are doing a word problem, for instance, and are trying to find something like, say, speed, then grab your calculator and find the decimal approximation of  $\sqrt{3}$ :

$$\sqrt{3} \approx 1.732050808$$

...and round to an appropriate number of decimal places, like "1.7 ft/sec". On the other hand, you may be solving a plain old math problem (with no practical application), in which case you will almost certainly want the "exact" answer, so you'll just leave the answer as " $\sqrt{3}$ ".

When you add  $x$ 's, you do it in the manner of  $2x + 3x = 5x$ . We do the same with radicals:

- **Simplify:**  $2\sqrt{3} + 3\sqrt{3}$ .

$$2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$$

Adding and subtracting radicals is similar in ways to adding and subtracting polynomial terms. Just as you can not combine  $2x$  and  $3y$  (because they are not "like terms"), so also you can not combine  $2\sqrt{3} + 3\sqrt{5}$ . You can not combine 2 and  $3x$  (because they are not "like terms"); likewise, you can not combine  $2 + 3\sqrt{3}$ .

You will generally have to "simplify" square root expressions. Some are easy to do:

- **Simplify:**  $\sqrt{3} + 4\sqrt{3}$ .

$$\sqrt{3} + 4\sqrt{3} = 1\sqrt{3} + 4\sqrt{3} = 5\sqrt{3}$$

Don't assume that expressions with unlike radicals cannot be simplified, however. It is possible that, after simplifying the radicals, the expression can indeed be simplified. For instance:

- **Simplify:**  $\sqrt{9} + \sqrt{25}$ .

$$\sqrt{9} + \sqrt{25} = 3 + 5$$

Here is an important property of square roots:

- **Write as the product of two radicals:**  $\sqrt{6}$ .

$$\sqrt{6} = \sqrt{2 \cdot 3} = \sqrt{2}\sqrt{3}$$

How was I able to rearrange the original radical like that? Because square roots are flexible with multiplication. You can factor the insides of a square root, and then split the square root according to the factors. Sometimes it helps to manipulate the multiplication in the other direction:

- **Simplify by writing as one radical:**  $\sqrt{6}\sqrt{15}\sqrt{10}$

$$\begin{aligned}\sqrt{6}\sqrt{15}\sqrt{10} &= \sqrt{6 \cdot 15 \cdot 10} \\ &= \sqrt{900} = \sqrt{30 \cdot 30} \\ &= 30\end{aligned}$$

Here is an example of how you can use this multiplication property to simplify radical expressions:

- **Simplify:**  $\sqrt{50} + \sqrt{8}$ .

$$\begin{aligned}\sqrt{50} + \sqrt{8} &= \sqrt{25 \cdot 2} + \sqrt{4 \cdot 2} \\ &= \sqrt{25}\sqrt{2} + \sqrt{4}\sqrt{2} \\ &= 5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}\end{aligned}$$

In general, how do you figure out what can "come out" of a square root? Factor the innards, and any factor that occurs in pairs can come out. For example:

- **Simplify:**  $\sqrt{75x^6y^3}$ .

$$\begin{aligned}\sqrt{75x^6y^3} &= \sqrt{3 \cdot 5 \cdot 5 \cdot \text{xxxxxx} \cdot \text{yy} \cdot \text{y}} \\ &= \sqrt{3 \cdot 5 \cdot 5 \cdot \text{xx} \cdot \text{xx} \cdot \text{xx} \cdot \text{yy} \cdot \text{y}} \\ &= 5 \cdot \text{xxx} \cdot \text{y} \cdot \sqrt{3 \cdot \text{y}} \\ &= 5x^3y\sqrt{3y}\end{aligned}$$

So far we have dealt only with square roots, but you would manipulate other radicals in much the same way. However, we need first to cover the notation for these other radicals.

Non-square-root radicals need a number to indicate which root is meant. By default, the simple radical symbol, " $\sqrt{\quad}$ ", is assumed to mean "the square root". This is similar to exponents: if you have " $x^2$ ", you know this means " $x$  squared"; if you have just plain old " $x$ ", you know that the exponent is "1". Whenever there is no exponent, the exponent is understood to be "1"; to indicate other powers, you insert a number. In the same way, when there is no number on the radical, the

radical is understood to be the square root; to indicate other roots, you insert a number, called the "index", inside that little "hook" on the front of the radical symbol. For instance:

a square (second) root is written as  $\sqrt{\quad}$

a cube (third) root is written as  $\sqrt[3]{\quad}$

a fourth root is written as  $\sqrt[4]{\quad}$

a fifth root is written as:  $\sqrt[5]{\quad}$

...and so on.

When writing a square root, you can put a teeny "2" in, if you like, but this would be considered non-standard notation.

The process of simplification in these other roots works similarly to simplification of square roots. If you have a cube root, you can take out any factor that occurs in threes:

- **Simplify:**  $\sqrt[3]{8}$ .

$$\sqrt[3]{8} = \sqrt[3]{2 \cdot 2 \cdot 2} = 2$$

- **Simplify:**  $\sqrt[3]{54}$ .

$$\sqrt[3]{54} = \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 2} = \sqrt[3]{3 \cdot 3 \cdot 3} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$$

In a fourth root, take out any factor that occurs in fours; in a fifth root, take out any factor that occurs in fives; etc.

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**Do not confuse "simplifying" with "solving".** If you have " $x^2 = 4$ ", and you take the square root of either side, you will get  $x = \pm 2$ , because you could square either of  $-2$  and  $+2$  to get  $+4$ . But  $\sqrt{4} \neq \pm 2$ , because " $\sqrt{4}$ " is specifically the *positive* square root. That is:

$$\sqrt{4} = 2$$

...and:

$$-\sqrt{4} = -2$$

Do not confuse these. Finding all the given roots of a number (such as finding all the square roots of 4 from the equation " $x^2 = 4$ ") is different from simplifying an expression (such as " $\sqrt{4}$ "), because the former uses all the roots but the latter uses only the "principal" root.

You have seen the simple case of radical multiplication, but you will probably have simplification problems that are more complex. Work with radicals in a manner similar to how you worked with polynomials. For instance, if you had to multiply  $3(x + 2)$ , you would take the 3 through the parentheses to get  $3x + 6$ . Similarly:

- **Simplify**  $3(2 + \sqrt{5})$

$$\begin{aligned} 3(2 + \sqrt{5}) &= 3(2) + 3(\sqrt{5}) \\ &= 6 + 3\sqrt{5} \end{aligned}$$

- **Simplify**  $\sqrt{3}(2 + \sqrt{5})$

$$\begin{aligned} \sqrt{3}(2 + \sqrt{5}) &= \sqrt{3}(2) + \sqrt{3}(\sqrt{5}) \\ &= 2\sqrt{3} + \sqrt{3 \cdot 5} \\ &= 2\sqrt{3} + \sqrt{15} \end{aligned}$$

- **Simplify**  $\sqrt{3}(2\sqrt{3} + \sqrt{5})$

$$\begin{aligned} \sqrt{3}(2\sqrt{3} + \sqrt{5}) &= \sqrt{3}(2\sqrt{3}) + \sqrt{3}(\sqrt{5}) \\ &= 2\sqrt{3 \cdot 3} + \sqrt{3 \cdot 5} \\ &= 2 \cdot 3 + \sqrt{15} = 6 + \sqrt{15} \end{aligned}$$

- **Simplify**  $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

- **Simplify**  $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

Note in the last example above how I ended up with all whole numbers. I had multiplied two radical "binomials" together and gotten an answer that contained no radicals. You may also have noticed that the two "binomials" were the same except for the sign in the middle. This is important.

Given  $\sqrt{a} + \sqrt{b}$ , the **conjugate** is  $\sqrt{a} - \sqrt{b}$ .

That is, the conjugate has the same numbers, but has the opposite sign in the middle. In other words, not only is  $\sqrt{a} - \sqrt{b}$  the conjugate of  $\sqrt{a} + \sqrt{b}$ , but  $\sqrt{a} + \sqrt{b}$  is the conjugate of  $\sqrt{a} - \sqrt{b}$ .

When you multiply conjugates, you are doing something similar to what happens with a difference of squares:

$$a^2 - b^2 = (a + b)(a - b)$$

When you multiply the factors  $a + b$  and  $a - b$ , the middle " $ab$ " terms cancel out:

The same thing happens when you multiply conjugates:

We will see shortly why this matters. To get to that point, let's take a look at fractions containing radicals in their denominators.

For whatever reason, it is generally considered "improper" to leave radicals in the denominator. (This is similar to the fuss regarding improper fractions versus mixed numbers: once you get far enough in math, radicals in the denominator probably won't matter any more, just like, now that you're in algebra, your teacher doesn't get upset about improper fractions any more, like your teacher did back in sixth grade.) The process of getting rid of radicals in the denominator is called "**rationalizing the denominator**".

- Simplify  $\frac{2}{\sqrt{3}}$ .

Now you may think "This looks pretty darn simple already!" But, by "simplify", in this case they mean "get rid of the radical in the denominator". To do this, I will need to turn the  $\sqrt{3}$  into something else. If you think back to when you were dealing with fractions, you would get a common denominator by multiplying non-common denominators by some useful number. For instance, if you needed to turn two-thirds into some kind of nineths, you would multiply by three-over-three:

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{3}{3} = \frac{6}{9}$$

You follow this same sort of procedure with rationalizing denominators. If you multiply  $\sqrt{3}$  by another  $\sqrt{3}$ , you'll get "3". So I'll follow the pattern with radicals as we used to use with fractions:

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

When rationalizing denominators, figure out what you need to have in order to be able to take a square out of the radical (or a cube, if you're dealing with a cube root, etc.), multiply the fraction, top and bottom, by this value, and simplify. Here's another example:

- Simplify  $\frac{2}{\sqrt{6}}$ .

This one works just like the previous example, except that I can do some further simplifying at the end:

$$\frac{2}{\sqrt{6}} = \frac{2}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$

It is always a good idea to check if your fraction can be simplified. Usually it can't, but be sure to check.

- **Simplify**  $\frac{2}{\sqrt{18x^2y}}$ .

This is an instance of where I'll have to simplify, then multiply, and then simplify again. The final answer doesn't look a whole lot "simpler" than what I started with, but it won't have radicals in the denominator, and that's what they're looking for when they say to "simplify".

$$\begin{aligned} \frac{2}{\sqrt{18x^2y}} &= \frac{2}{\sqrt{9 \cdot 2 \cdot x^2 \cdot y}} \\ &= \frac{2}{3x\sqrt{2y}} \\ &= \frac{2}{3x\sqrt{2y}} \cdot \frac{\sqrt{2y}}{\sqrt{2y}} \\ &= \frac{2\sqrt{2y}}{3x \cdot 2y} = \frac{2\sqrt{2y}}{6xy} \\ &= \frac{\sqrt{2y}}{3xy} \end{aligned}$$


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